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**WAVE PROPAGATION IN LINEAR, BILINEAR  
AND TRILINEAR ELASTIC BARS**

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## Abstract

This paper is concerned with the role of supplementary conditions, such as the entropy inequality at shock waves or kinetic relations at phase boundaries, in the selection of physically appropriate solutions to systems of quasilinear differential equations describing wave propagation. The differences in this respect among various materials are illustrated by contrasting the behavior of waves in linear, bilinear and trilinear elastic bars.

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**1. Introduction.** Among the distinctions between waves governed by the linear wave equation and those governed by quasilinear equations, one pertains to the identification of physically meaningful solutions through supplementary conditions such as the entropy inequality at discontinuities; see, for example, [1]. Indeed, the fact that such additional requirements, while unnecessary in linear problems, are needed in the nonlinear case has long been familiar from the theory of shock waves in classical gas dynamics [2]. More recently, continuum models of the dynamics of phase transitions have supplied examples of quasilinear systems for which uniqueness fails in the Cauchy problem, even *with* the entropy inequality in force; see [3-6].

In the present paper, we are concerned with wave propagation in one-dimensional elastic bars. For such bars, the nature of the relation  $\sigma = \hat{\sigma}(\gamma)$  between the stress  $\sigma$  and the strain  $\gamma$  controls the extent to which the fundamental field equations and jump conditions must be supplemented in order to secure uniqueness in initial value-problems. When the stress-strain relation is linear, say  $\sigma = \mu\gamma$  with  $\mu > 0$ , no additional information is needed. In contrast, in the presence of a *nonlinear* relation  $\sigma = \hat{\sigma}(\gamma)$ , uniqueness for initial value problems generally fails; if  $\hat{\sigma}(\gamma)$  increases monotonically with  $\gamma$  and the stress-strain curve is *either* strictly convex *or* strictly concave (the "genuinely nonlinear" case [1]), uniqueness is restored by imposing the entropy inequality at all strain discontinuities; see [1, 7]. For stress-strain relations of the kind used in simple continuum models of stress-induced phase transformations, the function  $\hat{\sigma}(\gamma)$  is *neither* monotonic in  $\gamma$  *nor* strictly convex or concave, and the entropy inequality at wave-fronts is insufficient for uniqueness; additional constitutive information describing the physics of the phase transition must be added to the model.

In order to provide an elementary illustration of the situation just described, we consider three cases: in the first of these,  $\hat{\sigma}(\gamma)$  is linear in  $\gamma$ , and elastic waves are governed by the linear wave equation in one space dimension. In the second case,  $\hat{\sigma}(\gamma)$  is bilinear, and the theory is qualitatively analogous to that for gas dynamics, but much simpler in detail. Finally, we consider a "trilinear" function  $\hat{\sigma}(\gamma)$ , corresponding to a material that can undergo stress-induced solid-solid phase transitions. In each case, we study the *Riemann problem*, the special version of the Cauchy problem that reveals the wave pattern through which a given discontinuity will evolve. For piecewise linear stress-strain relations, the Riemann problems of interest can be solved explicitly and globally, making clear the extent of the need for restrictions beyond those imposed by the fundamental differential equations and jump conditions.

**2. Basic equations.** Suppose a bar of unit cross-sectional area occupies the interval  $(-\infty, \infty)$  of the  $x$ -axis in the undeformed state. During a motion, the particle at  $x$  in the undeformed state is carried to  $x + u(x,t)$  at time  $t$ , where  $u(x,t)$  is the displacement. We assume that  $u$  is continuous with piecewise continuous first and second derivatives for  $-\infty < x < \infty$ ,  $t \geq 0$ . The strain  $\gamma(x,t)$  and particle velocity  $v(x,t)$  are defined by

$$\gamma = u_x, \quad v = u_t, \quad (2.1)$$

wherever the derivatives exist. Balance of momentum and compatibility of  $\gamma$ ,  $v$  require that

$$\sigma_x = \rho v_t, \quad (2.2)$$

$$v_x = \gamma_t, \quad (2.3)$$

at points where  $\sigma$ ,  $v$  and  $\gamma$  are smooth; here,  $\sigma(x,t)$  is the stress at  $(x,t)$ , and  $\rho$  is the constant mass density in the undeformed state. If either  $\gamma$  or  $v$  is discontinuous across a curve  $x = s(t)$  in the  $x$ ,  $t$ -plane, balance of momentum and the smoothness of  $u$  provide the following jump conditions:

$$\bar{\sigma} - \sigma = -\rho \dot{s} (\bar{v} - v), \quad (2.4)$$

$$\bar{v} - v = -\dot{s} (\bar{\gamma} - \gamma), \quad (2.5)$$

where  $\dot{s}(t)$  is the velocity of the moving strain discontinuity, and for any  $g(x,t)$  we have written  $\bar{g} \equiv g(s(t) \pm, t)$ .

For the elastic bar,

$$\sigma = \hat{\sigma}(\gamma) , \quad -1 < \gamma < \infty , \quad (2.6)$$

where  $\hat{\sigma}(\gamma)$ , assumed to be continuous with a piecewise continuous derivative, is the given stress response function of the material. The strain  $\gamma$  is restricted to the range  $(-1, +\infty)$  in order to assure that the mapping  $x \rightarrow x + u(x,t)$  is invertible for each  $t$ .

Suppose that, during a motion,  $\gamma$ ,  $v$  and  $\sigma$  are smooth on  $[x_1, x_2] \times [t_1, t_2]$  *except* for a single moving strain jump at  $x=s(t)$ . Let

$$W(\gamma) = \int_0^\gamma \hat{\sigma}(\gamma') d\gamma' , \quad \gamma > -1 , \quad (2.7)$$

be the stored energy per unit undeformed volume of the bar, so that the total mechanical energy  $E(t)$  at time  $t \in [t_1, t_2]$  associated with the piece of the bar under consideration is

$$E(t) = \int_{x_1}^{x_2} \left[ W(\gamma) + \frac{1}{2} \rho v^2 \right] dx . \quad (2.8)$$

A direct calculation using (2.1) - (2.8) shows that

$$\sigma(x_2,t)v(x_2,t) - \sigma(x_1,t)v(x_1,t) - \dot{E}(t) = f(t)\dot{s}(t) , \quad (2.9)$$

where the *driving force*  $f$  acting on the strain discontinuity is defined by

$$f = \hat{f}(\bar{\gamma}, \bar{\gamma}^+) \equiv \int_{\bar{\gamma}}^{\bar{\gamma}^+} \hat{\sigma}(\gamma) d\gamma - \frac{1}{2} (\bar{\sigma}^+ + \bar{\sigma})(\bar{\gamma}^+ - \bar{\gamma}) . \quad (2.10)$$

Here  $\bar{\gamma}^{\pm} = \bar{\gamma}(t) = \gamma(s(t) \pm, t)$  and  $\bar{\sigma}^{\pm} = \bar{\sigma}(t) = \sigma(s(t) \pm, t) = \hat{\sigma}(\bar{\gamma}(s(t) \pm, t))$  are the strains and stresses, respectively, on the two sides of the strain discontinuity. Geometrically,  $\hat{f}(\bar{\gamma}, \bar{\gamma}^+)$  may be interpreted as the difference between the area under the stress-strain curve between  $\bar{\gamma}$  and  $\bar{\gamma}^+$  and the area of the trapezoid determined by  $\bar{\gamma}, \bar{\gamma}^+, \bar{\sigma}$  and  $\bar{\sigma}^+$ .

The left side of (2.9) represents the excess of the rate of work of the external forces over the rate of change of mechanical energy. Although this excess vanishes for smooth strain fields (since  $f=0$ ) or for stationary discontinuities ( $\dot{s}=0$ ), this is not the case in general. The right side of (2.9), which is the rate of mechanical dissipation, may be viewed as the rate of "work" done by the driving force  $f$  in moving the discontinuity at a velocity  $\dot{s}$ . If the material is considered to be thermoelastic, and if for simplicity we make the unrealistic assumption that the motions considered take place isothermally, then as shown in [8], the second law of thermodynamics requires that

$$f(t)\dot{s}(t) \geq 0 . \quad (2.11)$$

The *entropy inequality* (2.11) must hold at all strain discontinuities and at all times; under isothermal conditions, it is equivalent to the assertion that the entropy of a particle cannot decrease as the particle crosses a strain discontinuity.

At the moving discontinuity  $x=s(t)$ , the jump conditions (2.4), (2.5) imply

$$\rho \dot{s}^2 = \frac{\hat{\sigma}(\bar{\gamma}^+) - \hat{\sigma}(\bar{\gamma})}{\bar{\gamma}^+ - \bar{\gamma}} , \quad (2.12)$$

$$[\hat{\sigma}(\bar{\gamma}^+) - \hat{\sigma}(\bar{\gamma})](\bar{\gamma}^+ - \bar{\gamma}) = \rho(\bar{v}^+ - \bar{v})^2 . \quad (2.13)$$

The right side of (2.12) is thus necessarily non-negative for any pair of strains  $\bar{\gamma}, \bar{\gamma}^+$  that can occur at a strain jump. Conversely, if  $\bar{\gamma}, \bar{\gamma}^+$  are numbers in  $(-1, \infty)$  such that the right side of (2.12) is non-negative, then it is possible to find numbers  $\bar{v}, \bar{v}^+$  and  $\dot{s}$  through (2.13) and (2.12) such that the jump conditions (2.4), (2.5) and the entropy inequality (2.11) are satisfied. Unless the value of  $f$  associated with  $\bar{\gamma}, \bar{\gamma}^+$  through (2.10) is zero, only one of the two values of  $\dot{s}$  determined by (2.12) will be consistent with (2.11). When  $f$  vanishes, both values of  $\dot{s}$  determined by (2.12) are consistent with (2.11), and the discontinuity may propagate in either direction.

**3. The Riemann problem.** In this special case of the Cauchy problem, we look for weak solutions  $\gamma, v$  of the system (2.2), (2.3), (2.6) on the upper half of the  $x, t$ -plane that satisfy the following initial conditions:

$$\gamma(x, 0+) = \begin{cases} \gamma_L, & -\infty < x < 0, \\ \gamma_R, & 0 < x < +\infty, \end{cases} \quad v(x, 0+) = \begin{cases} v_L, & -\infty < x < 0, \\ v_R, & 0 < x < +\infty, \end{cases} \quad (3.1)$$

where  $\gamma_L, \gamma_R, v_L$  and  $v_R$  are given constants, with  $\gamma_L > -1, \gamma_R > -1$ . Solutions  $\gamma, v$  must of course satisfy the jump conditions (2.4), (2.5).

The Riemann problem is invariant under the scale change  $x \rightarrow kx, t \rightarrow kt$ , where  $k$  is any constant. We seek solutions with this same invariance. It is easily shown that, if the stress-strain relation  $\sigma = \hat{\sigma}(\gamma)$  is piecewise linear, strain discontinuities in such solutions can occur only on rays issuing from the origin in the  $x, t$ -plane:  $x = \dot{s}t, \dot{s} = \text{constant}$ ; moreover, between such rays,  $\gamma$  and  $v$  must be constant. For such materials, "fans" cannot occur. Thus scale-invariant solutions to a Riemann problem for a piecewise linear material have the following form (Figure 1):

$$\gamma(x,t) = \gamma_j, \quad v(x,t) = v_j \quad \text{for } s_j t < x \leq s_{j+1} t, \quad j = 0, \dots, N, \quad (3.2)$$

where  $\gamma_j$ ,  $v_j$ ,  $s_j$  and  $N$  are constants, with  $N$  a non-negative integer, and  $\gamma_0 = \gamma_L$ ,  $\gamma_N = \gamma_R$ ,  $v_0 = v_L$ ,  $v_N = v_R$ ,  $s_0 = -\infty$ ,  $s_{N+1} = +\infty$ . The case  $N=0$ , which can occur only when  $\gamma_L = \gamma_R$ ,  $v_L = v_R$ , is trivial; we may therefore assume that  $N \geq 1$  in all that follows. The  $\gamma_j$ 's must all exceed  $-1$ , and one must also have  $\gamma_{j+1} \neq \gamma_j$  for  $j=0, \dots, N-1$ .

At each of the  $N$  discontinuities in (3.2), the jump conditions (2.4), (2.5) require that

$$\left. \begin{aligned} \dot{s}(\gamma_j - \gamma_{j-1}) &= -(v_j - v_{j-1}), \\ \hat{\sigma}(\gamma_j) - \hat{\sigma}(\gamma_{j-1}) &= -\rho \dot{s}(v_j - v_{j-1}), \end{aligned} \right\} \quad j = 1, \dots, N. \quad (3.3)$$

Finally, the entropy inequality (2.11) is to be imposed at each strain discontinuity.

**4. Linear materials.** When  $\hat{\sigma}(\gamma) = \mu\gamma$ , where  $\mu > 0$  is the modulus of elasticity, (2.12) reduces to  $\dot{s} = \pm c$ , where  $c = (\mu/\rho)^{1/2}$ . As a result, there can be at most two rays  $x = st$  bearing strain discontinuities in the solution to the Riemann problem, so that in (3.2), one has  $N=2$ . It is then a simple matter to enforce the jump conditions (3.3) and thence to find the unique solution to the Riemann problem for the linear material:

$$\left. \begin{aligned} \gamma &= \gamma_L, \quad v = v_L & \text{for } x < -ct, \\ \gamma &= h, \quad v = v_L - c\gamma_L + ch & \text{for } -ct < x < ct, \\ \gamma &= \gamma_R, \quad v = v_R & \text{for } x > ct, \end{aligned} \right\} \quad (4.1)$$

where the parameter  $h$  is given by



$$h = \frac{c\gamma_L + c\gamma_R + v_R - v_L}{2c} . \quad (4.2)$$

For the linear material, the driving force  $f$  of (2.10) vanishes identically in  $\bar{\gamma}, \bar{\gamma}^+$ , as is easily verified; the entropy inequality (2.12) is thus trivially satisfied and plays no role in this case.

**5. Bilinear materials.** Suppose now that the stress response function has the following bilinear form:

$$\hat{\sigma}(\gamma) = \begin{cases} \mu_1 \gamma & \text{for } -1 < \gamma \leq \gamma_M , \\ \mu_2 \gamma + (\mu_1 - \mu_2) \gamma_M & \text{for } \gamma \geq \gamma_M . \end{cases} \quad (5.1)$$

In this section, we show that the Riemann problem for the bilinear material has only one solution that conforms to the entropy inequality; we also show that uniqueness fails if the entropy inequality is *not* enforced.

A point  $(\gamma, \hat{\sigma}(\gamma))$  is said to be on *branch 1* of the stress-strain curve if  $-1 < \gamma < \gamma_M$ , on branch 2 if  $\gamma > \gamma_M$ . For definiteness, we assume that  $\mu_2 < \mu_1$ , so that the material softens in branch 2; the case  $\mu_2 > \mu_1$  can be discussed in an entirely similar way.

A propagating discontinuity with strains  $\bar{\gamma}, \bar{\gamma}^+$  is a *1,1-sound wave* if  $\bar{\gamma}$  and  $\bar{\gamma}^+$  are both branch-1 strains; a *2,2-sound wave* is defined analogously. If one of the strains  $\bar{\gamma}, \bar{\gamma}^+$  happens to have the value  $\gamma_M$  corresponding to the corner in the stress-strain curve, we call the discontinuity a *1,1-* or a *2,2-* sound wave according to whether the remaining strain is on branch 1 or branch 2. If  $\bar{\gamma}$  and  $\bar{\gamma}^+$  belong to branch 1 and branch 2, respectively, we speak of a *1,2-shock wave*; in a *2,1-shock wave*, the branches for  $\bar{\gamma}$  and  $\bar{\gamma}^+$  are reversed.

From (2.12), one finds that the velocity of a 1,1-sound wave is  $\pm c_1$ , where  $c_1 = (\mu_1/\rho)^{1/2}$ ; for a 2,2-sound wave, the velocity is  $\pm c_2$ , where  $c_2 = (\mu_2/\rho)^{1/2}$ . Note that  $c_1 > c_2$ , so that 1,1-sound waves travel faster than 2,2's. For shock waves, the velocity depends on the local strains; (2.12) yields:

$$\dot{s}^2 = \frac{c_2^2 \bar{\gamma}^+ - c_1^2 \bar{\gamma} + (c_1^2 - c_2^2) \gamma_M}{\bar{\gamma}^+ - \bar{\gamma}} \quad \text{for a 1,2-shock wave,} \quad (5.2)$$

$$\dot{s}^2 = \frac{c_1^2 \bar{\gamma}^+ - c_2^2 \bar{\gamma} - (c_1^2 - c_2^2) \gamma_M}{\bar{\gamma}^+ - \bar{\gamma}} \quad \text{for a 2,1-shock wave.} \quad (5.3)$$

It follows that for *either* type of shock wave,

$$c_2^2 < \dot{s}^2 < c_1^2, \quad (5.4)$$

so that the speed  $|\dot{s}|$  of a shock wave is always "intersonic".

For sound waves, the strains on either side of the jump belong to the same linear branch of the stress-strain curve; it follows from (2.10) that the driving force  $f$  at a sound wave of either type vanishes, and the entropy inequality (2.11) is trivially satisfied.

For a 1,2-shock wave, the area under the stress-strain curve between  $\bar{\gamma}$  and  $\bar{\gamma}^+$  always exceeds the area of the trapezoid determined by  $\bar{\gamma}$ ,  $\bar{\gamma}^+$ ,  $\hat{\sigma}(\bar{\gamma}^+)$  and  $\hat{\sigma}(\bar{\gamma})$  when  $\mu_2 < \mu_1$ , so the driving force  $f$  of (2.10) is always positive at such a wave. At a shock of 2,1-type,  $f$  is always negative. It then follows from (5.4) and the entropy inequality (2.11) that the velocity  $\dot{s}$  of a 1,2-shock wave is positive, while that of a 2,1-shock wave is negative; in view of (5.4), one concludes that

$$c_2 < \dot{s} < c_1 \text{ for a 1,2-shock wave. } -c_1 < \dot{s} < -c_2 \text{ for a 2,1-shock wave.} \quad (5.5)$$

We now show that, in any solution to the Riemann problem that conforms to the entropy inequality (2.11), there are at most two shock waves, and if there are two, one is of 2,1-type and has negative velocity, while the other is of 1,2-type and has positive velocity; see (5.5). Suppose there are at least three shock waves, and let  $x=\dot{s}_j t$ ,  $j=1,2,3$ , with  $\dot{s}_1 < \dot{s}_2 < \dot{s}_3$ , represent three consecutive ones. Note first that, by (5.5), either  $\dot{s}_1$  and  $\dot{s}_2$  are both negative, or  $\dot{s}_2$  and  $\dot{s}_3$  are both positive. Hence either there are two consecutive shocks of 2,1-type, or there are two consecutive 1,2-shocks. Since the strain must change branches across each shock, neither of these cases is possible, and so three shock waves can not occur in any solution of the Riemann problem that satisfies the entropy inequality at each of its strain discontinuities. Clearly, if there are *two* shocks in any such solution, their velocities cannot have the same sign, so that by (5.5), the shocks are of opposite types.

The fact that there can be at most two shocks in any scale-invariant solution to the Riemann problem that fulfills the entropy inequality at each strain jump makes it possible to exhibit *all* such solutions. We say that a Riemann problem (or the corresponding initial data) is of *p,q-type* if  $\gamma_L$  is on branch p,  $\gamma_R$  on branch q, and we consider the various possible cases.

*Case 1. 1,1-initial data.* Suppose first that the initial data in (3.1) are such that  $\gamma_L$  and  $\gamma_R$  are both on branch 1 of the stress-strain curve. Clearly the number of shock waves must be even in this case; in view of the general result established above, we conclude that the number of shock waves must be either zero or two. But with initial data of 1,1-type, a solution with shock waves would necessarily involve a 1,2-shock with negative propagation velocity and a 2,1-shock with positive velocity, contradicting (5.5). Thus all discontinuities in any scale-invariant solution to the 1,1-Riemann problem must be sound waves. More detailed consideration shows that there

are exactly two types of solutions available: in one, the pattern of strain discontinuities in the  $x, t$ -plane is as shown in Figure 2a. The only discontinuities are two 1,1-sound waves represented by the solid lines issuing from the origin in the figure. In the other, there are four sound waves, as illustrated in Figure 2b.

For solutions of the type described by Figure 2a, the jump conditions (3.3) at the two sound waves determine the unknown constant strain and particle velocity between the two discontinuities, leading to a strain field given by

$$\gamma(x, t) = \begin{cases} \gamma_L, & x < -c_1 t, \\ h, & -c_1 t < x < c_1 t, \\ \gamma_R, & x > c_1 t, \end{cases} \quad (5.6)$$

where the parameter  $h$  is now defined by

$$h = \frac{c_1 \gamma_L + c_1 \gamma_R + v_R - v_L}{2c_1}; \quad (5.7)$$

cf. (4.1), (4.2). (For brevity, we henceforth omit formulas for the particle velocity field when giving explicit solutions;  $v(x, t)$  is easily found once the strain field is known.) The strain as given by (5.6) must lie in the branch-1 interval  $(-1, \gamma_M)$ ; this will be the case if and only if the given data are such that  $h < \gamma_M$ . Hence (5.6), (5.7), together with the appropriate particle velocity field, provide a solution to the 1,1-Riemann problem satisfying the entropy inequality if and only if this condition holds.

When  $h > \gamma_M$ , one can show that the appropriate pattern of discontinuities is that shown in Figure 2b. The jump conditions (3.3) now lead to a strain field given by

$$\gamma(x, t) = \begin{cases} \gamma_L, & x < -c_1 t, \\ \gamma_M, & -c_1 t < x < -c_2 t, \\ \gamma_M + \frac{c_1}{c_2} (h - \gamma_M), & -c_2 t < x < c_2 t, \\ \gamma_M, & c_2 t < x < c_1 t, \\ \gamma_R, & x > c_1 t, \end{cases} \quad (5.8)$$

with  $h$  again expressed in terms of initial data by (5.7).

Since the only discontinuities in the solutions (5.6), (5.8) are sound waves and the material is bilinear, the entropy inequality is trivially satisfied at each jump. It should be noted, however, that the latter inequality plays a crucial role in reaching the conclusion that the wave pattern is as shown in Figure 2.

When  $h \rightarrow \gamma_M^-$  in (5.6) or  $h \rightarrow \gamma_M^+$  in (5.8), the two limiting strain fields coincide, as do the corresponding particle velocities. The limiting fields represent entropically admissible solutions to the 1,1-Riemann problem when  $h = \gamma_M$ .

Thus a unique solution to the 1,1-Riemann problem that satisfies the entropy inequality exists for every choice of initial data; for data such that  $h < \gamma_M$ , it is given by (5.6), for  $h > \gamma_M$  by (5.8). For  $h < \gamma_M$ , the solution (5.6) leaves the bar entirely on branch 1 of the stress-strain curve for all time, whereas for  $h > \gamma_M$ , the bar ultimately changes from branch 1, where it was initially, to branch 2.

*Case 2. 1,2-initial data.* Consider now a Riemann problem whose initial data are of 1,2-type:  $\gamma_L$  on branch 1,  $\gamma_R$  on branch 2. When applied to this case, the general results above imply that there can be no more than one shock wave. Indeed, with  $h$  now defined to be

$$h = \frac{c_1 \gamma_L + c_2 \gamma_R + v_R - v_L}{c_1 + c_2}, \quad (5.9)$$

one readily shows that when the initial data are such that  $h < \gamma_M$ , the only scale-invariant solution is one whose discontinuities are as shown in Figure 3a. By analyzing the jump conditions (3.3) at the shock wave  $x = \dot{s}t$  and at the sound wave  $x = -c_2 t$ , one finds that the velocity  $\dot{s}$  of the shock wave is given in terms of initial data by

$$\dot{s} = \frac{(c_1 - c_2)\gamma_M + c_2 \gamma_R - c_1 h}{\gamma_R - h}; \quad (5.10)$$

the strain field associated with this solution is found to be

$$\gamma(x, t) = \begin{cases} \gamma_L, & x < -c_1 t, \\ \frac{(c_1 + c_2)h + (\dot{s} - c_2)\gamma_R}{c_1 + \dot{s}}, & -c_1 t < x < \dot{s}t, \\ \gamma_R, & x > \dot{s}t. \end{cases} \quad (5.11)$$

One can show that the strain between the sound wave and the shock wave lies on branch 1 of the stress-strain curve if and only if the velocity  $\dot{s}$  of the 1,2-shock wave lies in the interval  $c_2 < \dot{s} < c_1$ , or equivalently if and only if  $h < \gamma_M$ . The entropy inequality (2.11) holds because  $\dot{s} > 0$  and  $f > 0$  at a 1,2-shock wave.

When  $h > \gamma_M$ , the solution can be shown to have the form shown in Figure 3b: there are three sound waves and no shock waves. Detailed calculation gives the strain field as

$$\gamma(x, t) = \begin{cases} \gamma_L, & x < -c_1 t, \\ \gamma_M, & -c_1 t < x < -c_2 t, \\ \frac{c_1 + c_2}{2 c_2} h - \frac{c_1 - c_2}{2 c_2} \gamma_M, & -c_2 t < x < c_2 t, \\ \gamma_R, & x > c_2 t. \end{cases} \quad (5.12)$$

When  $h \rightarrow \gamma_M^-$ , the solution represented by (5.11) tends to that given by (5.12) when  $h \rightarrow \gamma_M^+$ .

As in the case of 1,1-initial data, the 1,2-Riemann problem has a unique solution conforming to the entropy inequality at shock waves for every set of initial data. When the data are such that  $h < \gamma_M$ , the solution is given by (5.11), and at infinite time, the strains in the bar are all on branch 1. If  $h > \gamma_M$ , the solution is given by (5.12), and the bar is ultimately on branch 2. The case  $h = \gamma_M$  is again treated by taking the appropriate limit of (5.11) or (5.12).

*Case 3. 2,2-initial data.* Finally, for the Riemann problem with 2,2-initial data, there are again two types of entropically admissible solutions corresponding to two subclasses of initial data delineated by a parameter  $h$  that is now defined by

$$h = \frac{c_2 \gamma_L + c_2 \gamma_R + v_R - v_L}{2 c_2}. \quad (5.13)$$

If  $h < \gamma_M$ , the solution has the structure illustrated in Figure 4a; it involves two shock waves and no sound waves, and it causes the bar to change from branch 2 initially to branch 1 finally. If

$h > \gamma_M$ , the solution has no shocks and two 2,2-sound waves, as in Figure 4b; it leaves the bar on branch 2. Since the analytic details are complicated in the 2,2-case, we omit the discussion of the form taken by the strain field. It is important to note, however, that the velocities  $\dot{s}_1$  and  $\dot{s}_2$  of the two shock waves in the case  $h < \gamma_M$  are fully determined in terms of initial data.

Thus in summary, with the entropy inequality in force, one finds that the Riemann problem for the bilinear material has a unique scale-invariant solution for every set of initial data. To demonstrate that this *fails* to be true if the entropy inequality is relinquished, we reconsider the 1,2-Riemann problem. If one seeks a solution whose strain discontinuities are as shown in Figure 5, one is led to the strain field given by

$$\gamma(x,t) = \begin{cases} \gamma_L, & x < -c_1 t, \\ \frac{c_2 + \dot{s}}{c_1 + \dot{s}} h + \frac{c_1 - c_2}{c_1 + \dot{s}} \gamma_M, & -c_1 t < x < \dot{s} t, \\ \frac{c_1 - \dot{s}}{c_2 - \dot{s}} h - \frac{c_1 - c_2}{c_2 - \dot{s}} \gamma_M, & \dot{s} t < x < c_2 t, \\ \gamma_R, & x > c_2 t, \end{cases} \quad (5.14)$$

where  $h$  is given by (5.9). If  $h > \gamma_M$ , the strain field given by (5.14) takes values in the appropriate intervals, and together with a suitable particle velocity field, satisfies all jump conditions, the initial conditions, and the differential equations, *for any value of the shock wave velocity  $\dot{s}$  in the interval  $(-c_1, -c_2)$* . Thus one can construct a *one-parameter family* of solutions to the 1,2-Riemann problem for  $h > \gamma_M$ , none of which coincides with the solution represented by (5.12). However, in the field (5.14), the shock wave is of 1,2-type, and it travels with a negative velocity. This violates the entropy inequality, and hence invalidates the one-parameter family of solutions with strain fields (5.14).



**6. Trilinear materials and phase transitions.** We turn now to the *trilinear* elastic material, whose stress response function is given by

$$\hat{\sigma}(\gamma) = \begin{cases} \mu_1 \gamma, & -1 < \gamma \leq \gamma_M, \\ -\mu_2 \gamma + \sigma_2, & \gamma_M \leq \gamma \leq \gamma_m, \\ \mu_3 \gamma, & \gamma \geq \gamma_m; \end{cases} \quad (6.1)$$

the graph of  $\hat{\sigma}(\gamma)$  is shown in Figure 6. It is assumed that  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are all positive, so that in particular, the second branch of the curve has negative slope. We identify each branch of the curve with a *phase* of the material: branches 1 and 3 are associated with stable or metastable phases, while the declining branch corresponds to an unstable phase. As a function of  $\gamma$  for fixed  $\sigma$ , the potential defined by  $G(\gamma, \sigma) = W(\gamma) - \sigma\gamma$ , with  $W$  given by (2.7), has extrema where  $\sigma = W'(\gamma) = \hat{\sigma}(\gamma)$ . When  $\hat{\sigma}(\gamma)$  is given by (6.1) and for  $\sigma$  between  $\sigma_m$  and  $\sigma_M$  (Figure 6),  $G(\cdot, \sigma)$  exemplifies a "two-well potential", having two minima corresponding to the stable and metastable phases separated by a maximum that corresponds to the unstable phase.

In this section, we shall cite results from [3] that show that the Riemann problem for the trilinear material does *not* have a unique solution, even with the entropy inequality in force. We also indicate how the breakdown in uniqueness can be repaired.

A strain discontinuity is said to be a 1,1- or 3,3-sound wave if the strains  $\bar{\gamma}, \bar{\gamma}^+$  on either side are both on branch 1 or both on branch 3, respectively. By (2.12), the negative slope of branch 2 implies that there are no sound waves of 2,2-type. If  $\bar{\gamma}$  is on branch 1 and  $\bar{\gamma}^+$  is on branch 3, we call the discontinuity a 1,3-*phase boundary*, rather than a 1,3-shock wave. A 3,1-phase boundary is defined analogously. According to (2.12), the velocities of 1,1- and

3,3-sound waves are  $\pm c_1$  and  $\pm c_3$ , respectively, where  $c_1=(\mu_1/\rho)^{1/2}$ ,  $c_3=(\mu_3/\rho)^{1/2}$ , with  $c_3 < c_1$ . By (2.12), the value of  $\dot{s}^2$  coincides with the slope of the chord joining the two points on the stress-strain curve that correspond to the states on the two sides of the discontinuity. Exploiting this geometric result leads to the conclusion that the velocity  $\dot{s}$  of a phase boundary of either 1,3- or 3,1-type satisfies

$$\dot{s}^2 < c_*^2, \quad (6.2)$$

where  $c_* = \{(c_1^2 + \gamma_m c_3^2)/(1 + \gamma_m)\}^{1/2}$ ; one has  $c_3 < c_* < c_1$ . A phase boundary is said to be *subsonic* if  $|\dot{s}| < c_3$ , *intersonic* if  $c_3 < |\dot{s}| < c_*$ . It is important to note that the velocity  $\dot{s}$  of a phase boundary may be zero, corresponding to a stationary interface between two phases in equilibrium.

As in the case of bilinear materials, the driving force vanishes at a sound wave of either 1,1- or 3,3-type. At a phase boundary, however, the driving force  $f$  coincides with the signed difference between the area under the stress-strain curve between  $\bar{\gamma}$  and  $\bar{\gamma}^\dagger$  and the area of the trapezoid determined by  $\bar{\gamma}$ ,  $\bar{\gamma}^\dagger$ ,  $\hat{\sigma}(\bar{\gamma}^\dagger)$  and  $\hat{\sigma}(\bar{\gamma})$ , so that the value of  $f$  may be positive, negative or zero. In view of (2.12), this means that a phase boundary may move to the right, to the left or in either direction according to whether  $f > 0$ ,  $f < 0$  or  $f = 0$ .

We now consider the Riemann problem for the trilinear material. Scale-invariant solutions of this problem must again have the form (3.2), and they are of course again subject to the jump conditions (3.3) and the entropy inequality (2.11) at each discontinuity. In the initial conditions (3.1), we confine attention to *metastable* initial data: data for which *neither* of the strains  $\gamma_L, \gamma_R$  belongs to branch 2 of the stress-strain curve (the unstable phase). All results stated without proof below are established in [3].

For the trilinear material, every scale-invariant, entropically admissible solution to a given Riemann problem with metastable initial data has the following properties:

- (i) no strain  $\gamma_j$  in (3.2) belongs to branch 2;
- (ii) there are at most two subsonic phase boundaries; if there are two, one moves with non-negative velocity, the other with non-positive velocity;
- (iii) there are at most two intersonic phase boundaries; if there are two, one moves with positive velocity, the other with negative velocity;
- (iv) either all phase boundaries are subsonic, or all are intersonic.

The proofs of these results rely heavily on the entropy inequality (2.11).

For the Riemann problem with metastable initial data, the qualitative conclusions stated above make it possible to construct explicitly *all* solutions that satisfy the entropy inequality; this was done in [3]. One finds that the situation regarding uniqueness differs drastically from that encountered for the bilinear material, for which - as we have shown in the preceding section - the entropy inequality is always sufficient for uniqueness. Indeed, two new phenomena arise in the trilinear case. To illustrate the first of these, it is sufficient to consider the case of *1,3-initial data* in (3.1):  $\gamma_L$  on branch 1,  $\gamma_R$  on branch 3. Let

$$h = \frac{c_1 \gamma_L + c_3 \gamma_R + v_R - v_L}{c_1 + c_3} ; \quad (6.3)$$

$h$  depends only on initial data. As shown in [3], if the initial data are such that  $h > 0$ , every solution of the 1,3-Riemann problem that satisfies the entropy inequality has the structure

sketched in Figure 7a: there is a single *subsonic* phase boundary at  $x=\dot{s}t$  and, in general, there are two sound waves, one of 1,1-type, the other 3,3. The strain field in every such solution depends on  $\dot{s}$  and on the initial data only through  $h$ . The new feature is that *the velocity  $\dot{s}$  of the 1,3-phase boundary is not determined by the initial data*. The requirement that the strains in the various sectors of the  $x,t$ -plane must lie on the appropriate branches of the stress-strain curve reduces to

$$\gamma_m \frac{c_3 - \dot{s}}{c_1 - \dot{s}} \leq h \leq \gamma_M \frac{c_1 + \dot{s}}{c_3 + \dot{s}}, \quad -c_3 < \dot{s} < c_3. \quad (6.4)$$

Thus for given initial data such that  $h > 0$ , there is a one-parameter family (parameter  $\dot{s}$ ) of solutions of the Riemann problem, each with the discontinuity structure shown in Figure 7a. The inequalities (6.4) correspond to the region in the  $\dot{s}, h$ -plane between the curves  $C_1$  and  $C_2$  shown in Figure 8. Each point  $(\dot{s}, h)$  in this region corresponds to a solution of the 1,3-Riemann problem for initial data giving rise to the specified  $h$  and for the specified  $\dot{s}$ .

Between  $C_1$  and  $C_2$ , there is a curve  $M$  whose points correspond to solutions in which the driving force  $f$  at the phase boundary vanishes. On one side of  $M$ ,  $f$  is positive, on the other side negative; therefore only those points in the hatched regions in Figure 8 correspond to solutions of the Riemann problem that satisfy the entropy inequality (2.11). Thus for given initial data such that  $h > 0$ , there is a one-parameter family (parameter  $\dot{s}$ ) of solutions of the Riemann problem, each with the discontinuity structure shown in Figure 7a and each satisfying the entropy inequality. In the  $\dot{s}, h$ -plane of Figure 8, this family corresponds to the horizontal line segment through the value of  $h$  determined by the initial data, the segment commencing on the curve  $M$ . The entropy inequality, while sufficient for uniqueness in the Riemann problem for the bilinear material, is thus insufficient in general for uniqueness in the same problem for the trilinear material.

As argued in [3], the lack of uniqueness just described arises from the need to specify additional constitutive information pertaining to the *kinetics* of the phase transition between branches 1 and 3 of the stress-strain curve. As suggested by the discussions in [3], [8], [9] and [11], such a kinetic relation might be expected to take the form of a relation  $f=\phi(\dot{s})$  between driving force  $f$  and phase boundary velocity  $\dot{s}$ , to be imposed in addition to the jump conditions at phase boundaries. In the context of the Riemann problem, such a kinetic law turns out to yield a materially-determined relation  $h=\hat{h}(\dot{s})$  between the datum  $h$  and the phase boundary velocity  $\dot{s}$ , corresponding to the curve  $K$  shown schematically in Figure 7, in which  $\hat{h}(\dot{s})$  increases monotonically with  $\dot{s}$ . A value of  $h$  corresponding to given initial data thus leads to a uniquely determined phase boundary velocity  $\dot{s}$  in the appropriate range, and hence to a unique solution of the Riemann problem that satisfies the entropy inequality.

The description above pertains to the 1,3-Riemann problem with initial data for which  $h$  in (6.3) is positive. It is shown in [3] that, when  $h$  is negative, the Riemann problem has a *unique* solution whose discontinuity pattern is as shown in Figure 7b. In contrast to the case for  $h > 0$ , the phase boundary is now intersonic, its velocity is determined by the initial data, and it is neither necessary nor possible to impose a kinetic relation. The difference between subsonically moving phase boundaries, for which kinetics must be prescribed, and intersonic ones, where kinetics may *not* be imposed, apparently has a parallel in combustion theory, where deflagrations require the prescription of the relevant kinetics, while detonations do not [10].

The second major distinction between the bilinear and trilinear materials pertains to the *nucleation* of the pertinent phase transition in the latter case. To indicate the sense in which this issue arises for the trilinear material, we consider the Riemann problem for initial data of *1,1-type*, so that the strains  $\gamma_L$  and  $\gamma_R$  are both in phase 1. Let

$$h = \frac{c_1 \gamma_L + c_1 \gamma_R + v_R - v_L}{2c_1} . \quad (6.5)$$

As shown in [3], the general consequences (i)-(iv) of the entropy inequality stated above imply that every entropically admissible solution to the 1,1-Riemann problem must conform to one of the two patterns of discontinuities shown in Figure 9. Solutions of the type associated with Figure 9a involve no phase boundaries and therefore no phase changes; for these, the strains in the bar always remain on branch 1 of the stress-strain curve. On the other hand, solutions of the type represented by Figure 9b involve a change of phase initiated at time  $t=0$  that propagates outward subsonically, leaving the bar in the high-strain phase (branch 3) in the long-time limit. The crucial fact pertaining to the 1,1-Riemann problem is that *there is an interval of values of the parameter  $h$ , and hence a non-empty set of initial data, for which solutions of both types exist*. A criterion is needed to select which of these two types of solutions is preferred when the initial data are such that both types are available; clearly, such a criterion serves precisely to determine whether the bar is to change phase or not. One such "nucleation criterion" is proposed in [3, 9]: a spontaneous phase transition takes place when the driving force  $f$  at an incipient phase boundary equals or exceeds a certain materially-determined critical level  $f_*$ . In the context of the 1,1-Riemann problem, this criterion turns out to select the solution of Figure 9a (no phase change) for  $h < h_*$ , where  $h_*$  is a critical value of  $h$  corresponding to  $f_*$ . For  $h \geq h_*$ , the solution must have discontinuities as in Figure 9b, and a phase transformation must occur. When the initial data give rise to a phase change, the phase boundary velocities  $\dot{s}$  and  $\dot{s}_*$  of Figure 9b are *not* determined by the initial data alone, but rather require the kinetic relation as well for their unique determination. In contrast, the solution of the type shown in Figure 9a involves no phase transition and is fully and uniquely determined by the initial data.

One can also consider a trilinear material for which the slope of branch 2 of the stress-strain curve remains positive but, say, less than the slopes of both branch 1 and branch 3.

In this case, stress is a monotonically increasing but neither convex nor concave function of strain. For such a material, equilibrium mixtures of phases and their associated stationary phase boundaries cannot occur, but lack of uniqueness in the Riemann problem of the kind that requires the specification of additional constitutive information nevertheless arises, just as in the case of the non-monotonic trilinear material treated above and described in Figure 1.

**7. Concluding remarks.** In the framework of the one-dimensional dynamical theory of bars, piecewise-linear elastic materials provide a vehicle for the illustration of the effect of the character of the stress-strain curve. While initial data alone determine a unique solution to the Riemann problem in the linear case, for the bilinear material, explicit calculations allow one to show that uniqueness for the Riemann problem fails unless the entropy inequality is imposed at all strain discontinuities, and that uniqueness holds globally when the inequality is in force. On the other hand, for the trilinear material, the entropy inequality is not enough to furnish uniqueness in the Riemann problem. From a physical viewpoint, this is perhaps to be expected, since the trilinear material may be thought of as a simple model for a material capable of undergoing stress-induced phase transitions. When one supplies pertinent information pertaining to the nucleation and the kinetics of the transition, uniqueness for the Riemann problem is again secured.

While the dynamical system associated with the bilinear material material does not literally qualify as a "genuinely nonlinear" system in the sense of Lax [1], it does mimic the properties of such systems. The trilinear material illustrates the nature of systems that are *not* genuinely nonlinear, whether they be hyperbolic or of mixed type.

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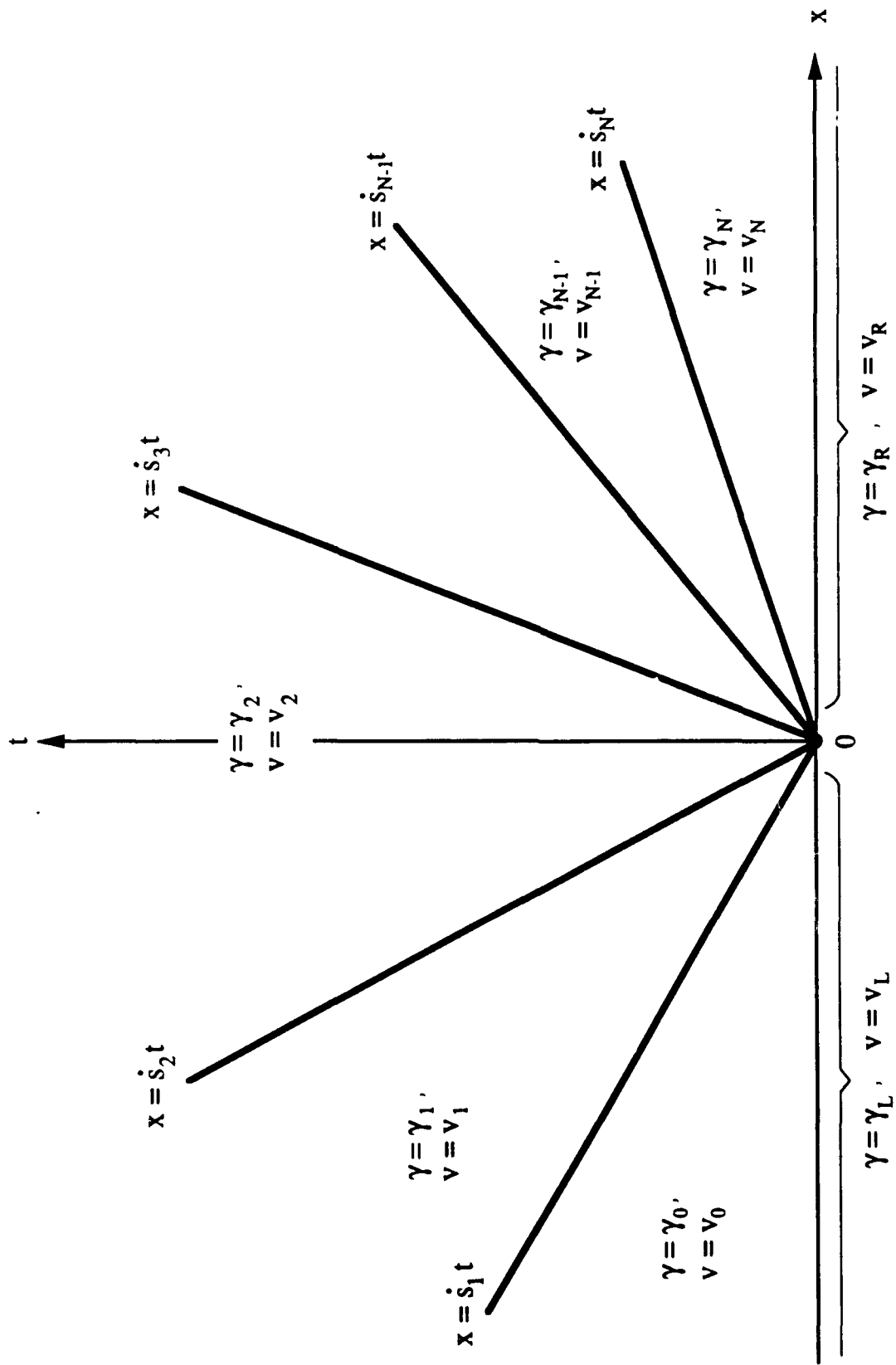


FIGURE 1. FORM OF SCALE - INVARIANT SOLUTIONS TO THE RIEMANN PROBLEM .

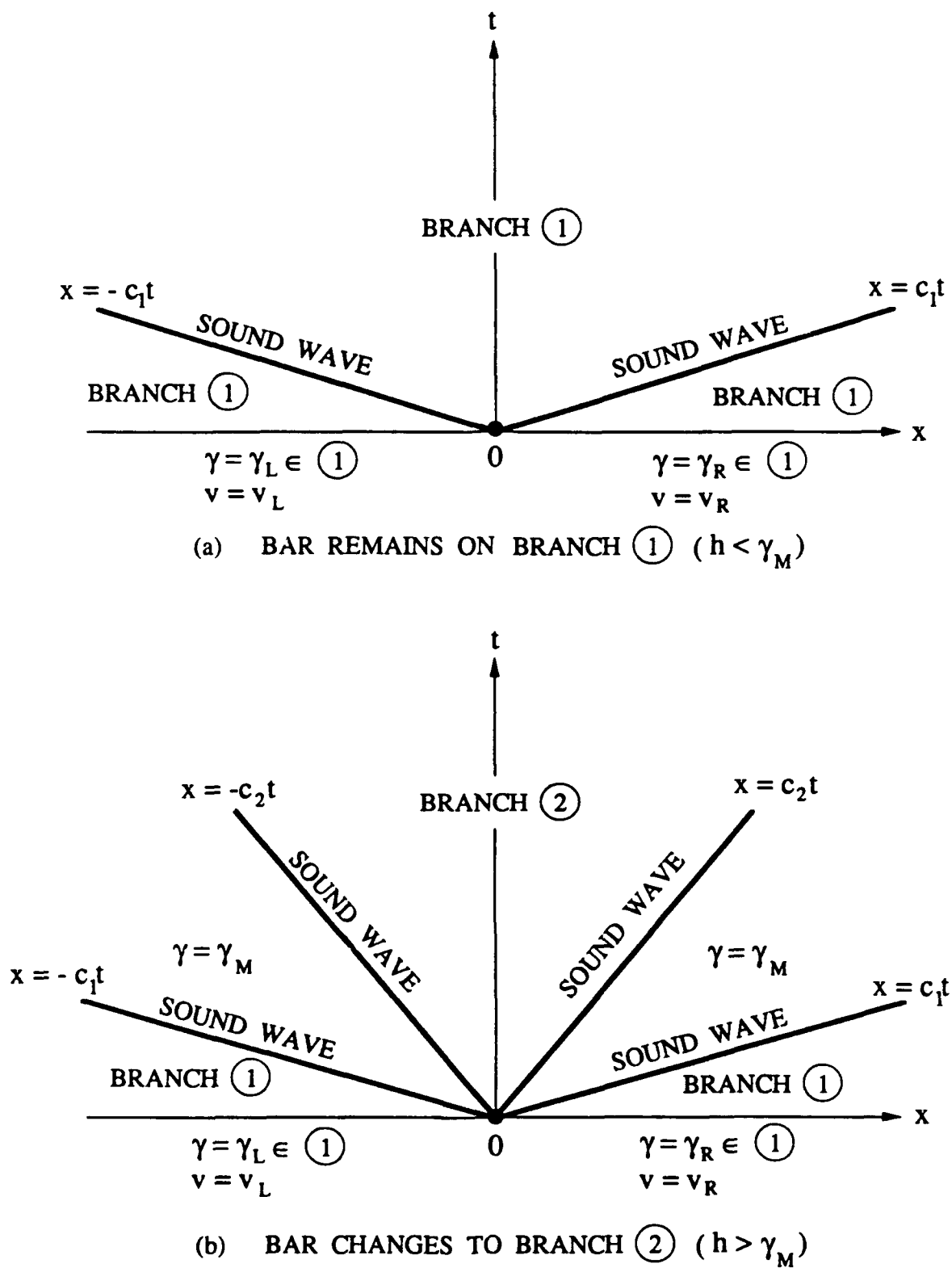
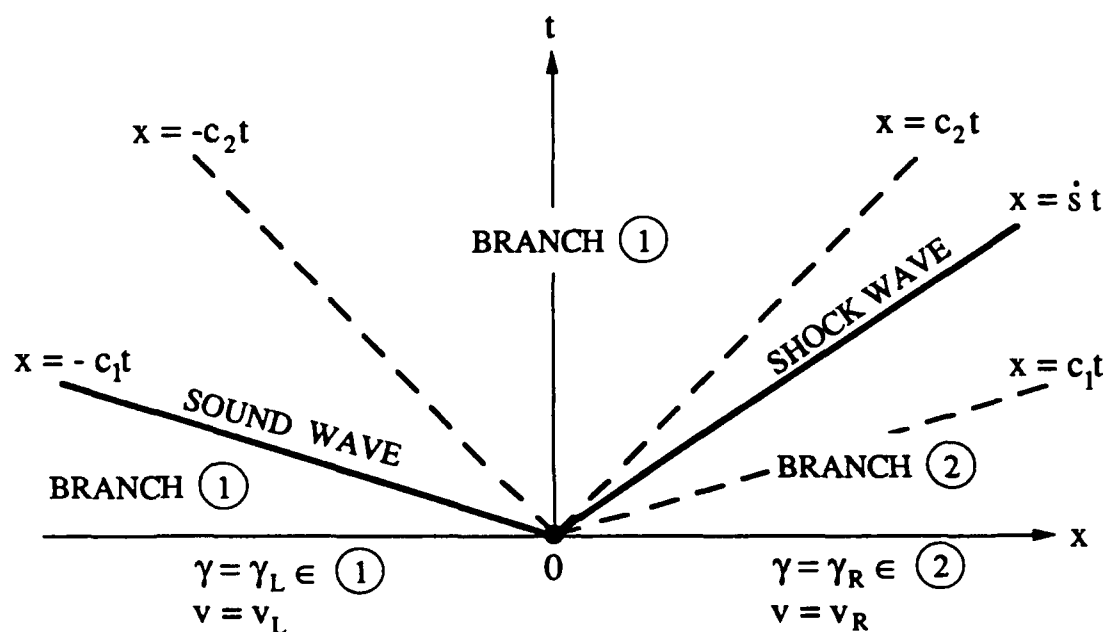
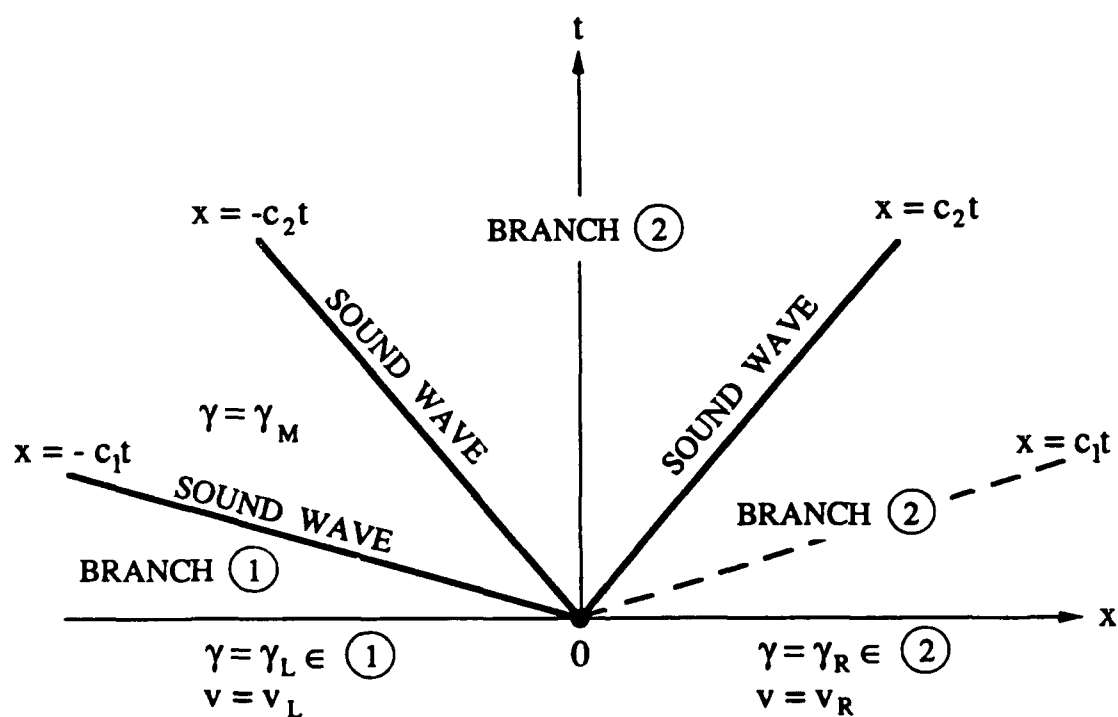


FIGURE 2. FORM OF SOLUTIONS TO THE 1,1-RIEMANN PROBLEM FOR THE BILINEAR MATERIAL.



(a) BAR ULTIMATELY ON BRANCH (1) ( $h < \gamma_M$ )



(b) BAR ULTIMATELY ON BRANCH (2) ( $h > \gamma_M$ )

FIGURE 3. FORM OF SOLUTIONS TO THE 1,2 - RIEMANN PROBLEM FOR THE BILINEAR MATERIAL.

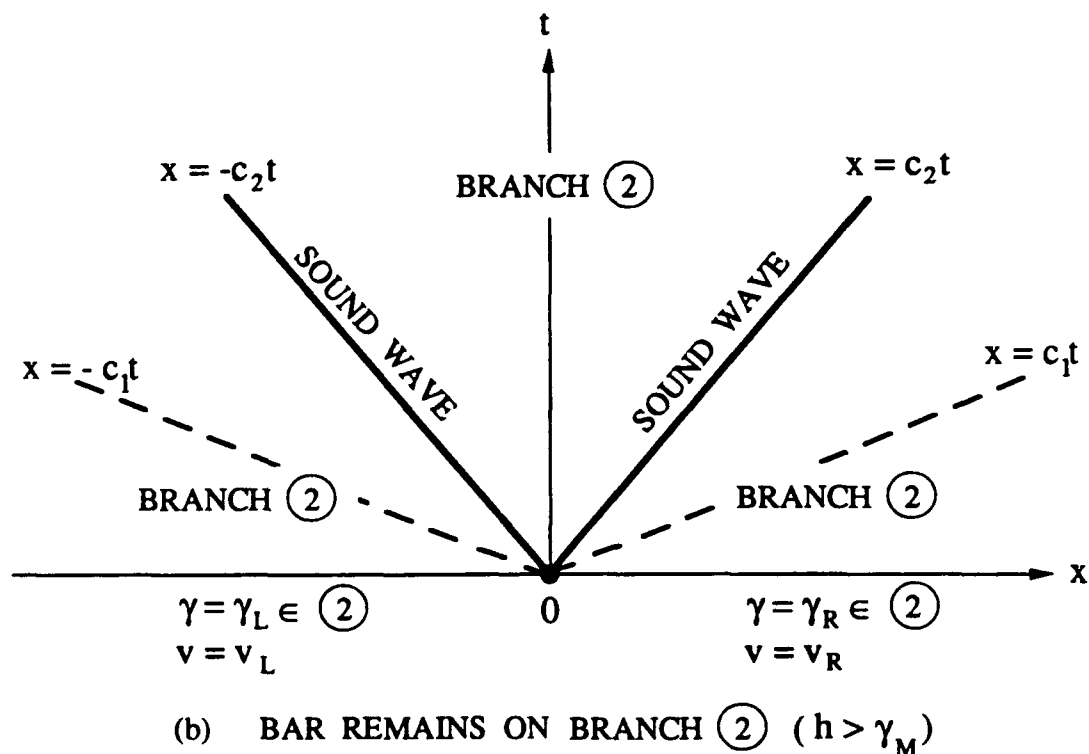
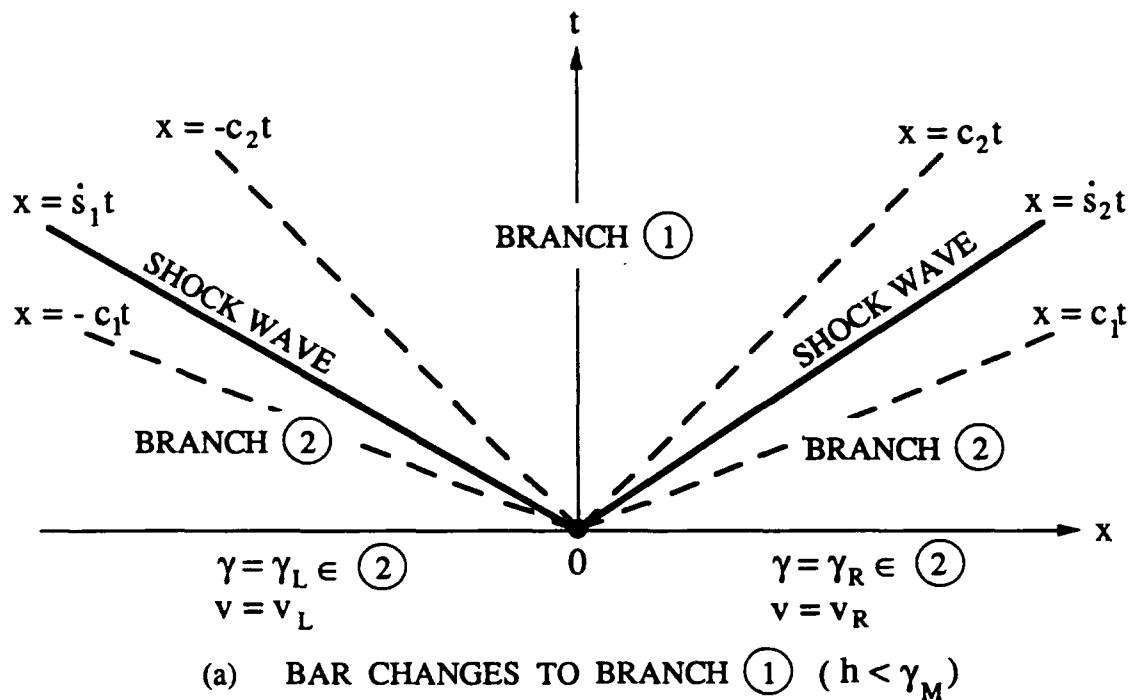


FIGURE 4. FORM OF SOLUTIONS TO THE 2,2 - RIEMANN PROBLEM FOR THE BILINEAR MATERIAL.

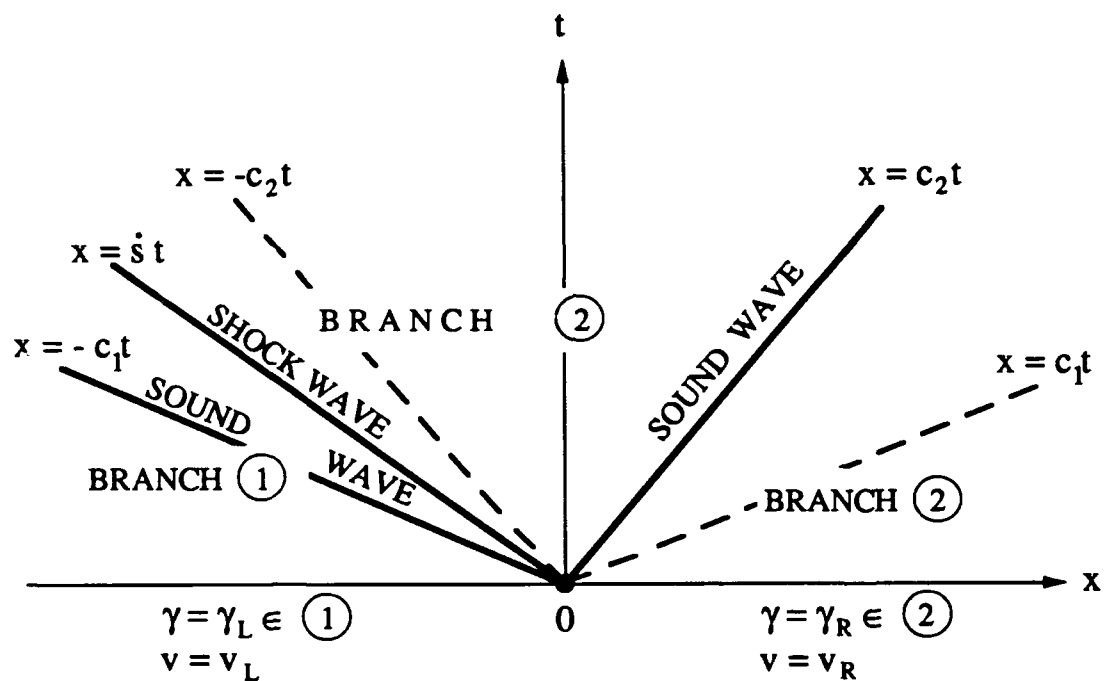


FIGURE 5. ENTROPICALLY INADMISSIBLE SOLUTION TO THE 1,2-RIEMANN PROBLEM FOR THE BILINEAR MATERIAL ( $h > \gamma_M$ ).

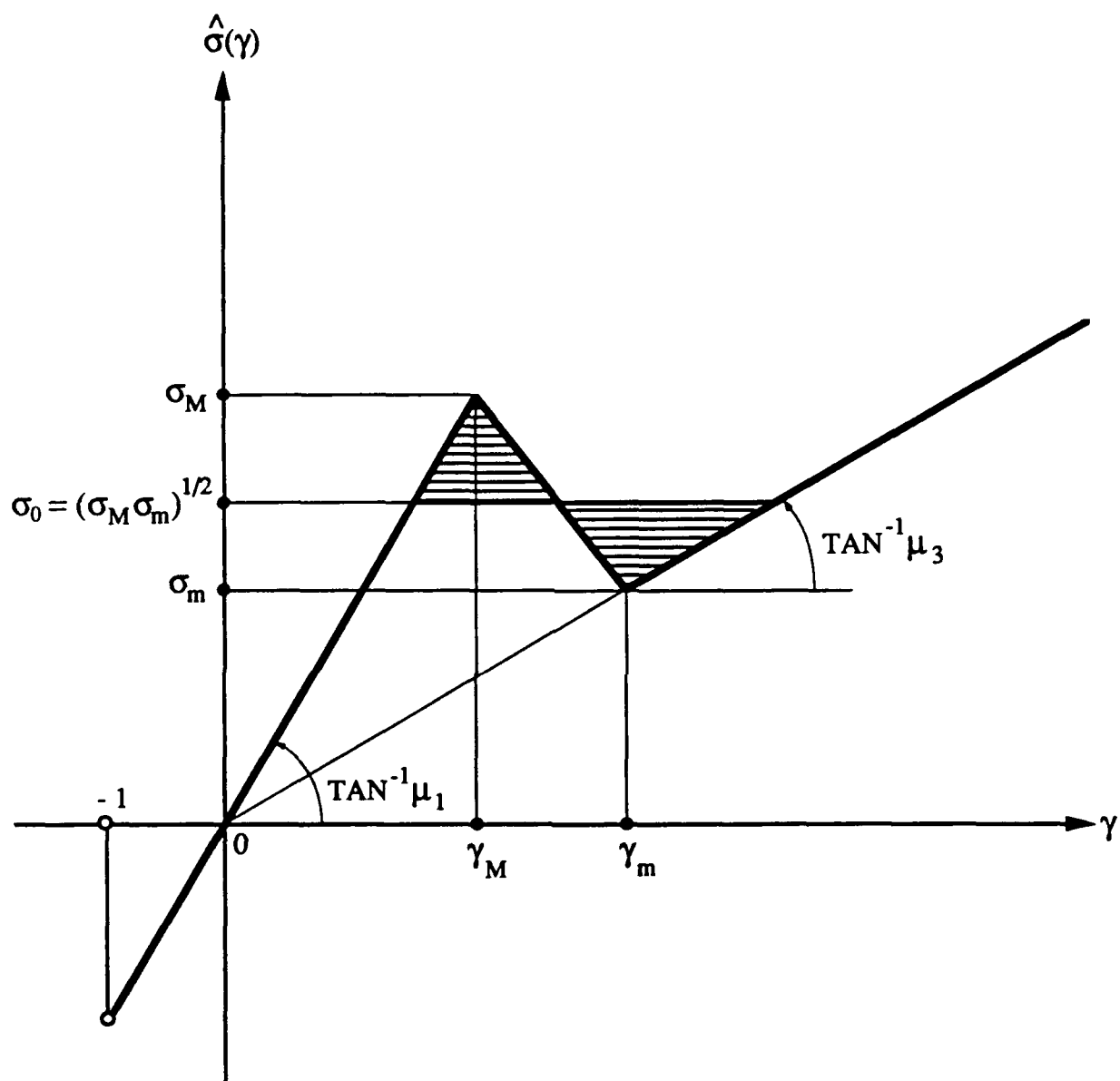


FIGURE 6. STRESS-STRAIN CURVE FOR THE TRILINEAR MATERIAL .

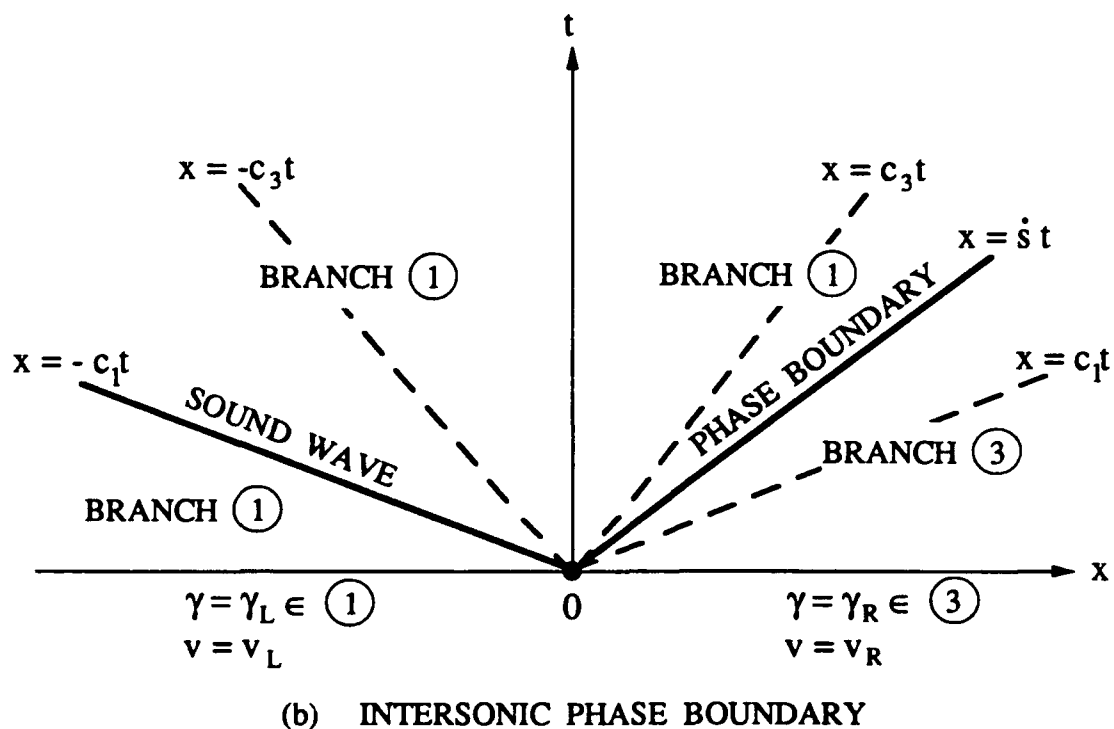
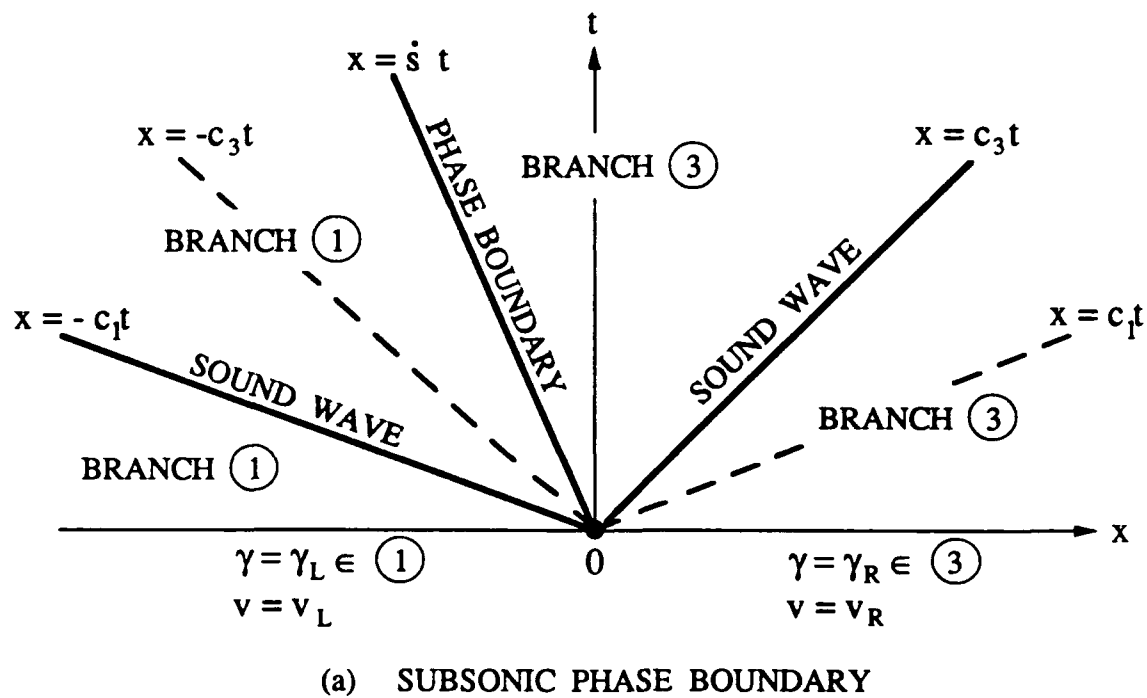


FIGURE 7. FORM OF SOLUTIONS TO THE 1, 3 - RIEMANN PROBLEM FOR THE TRILINEAR MATERIAL.

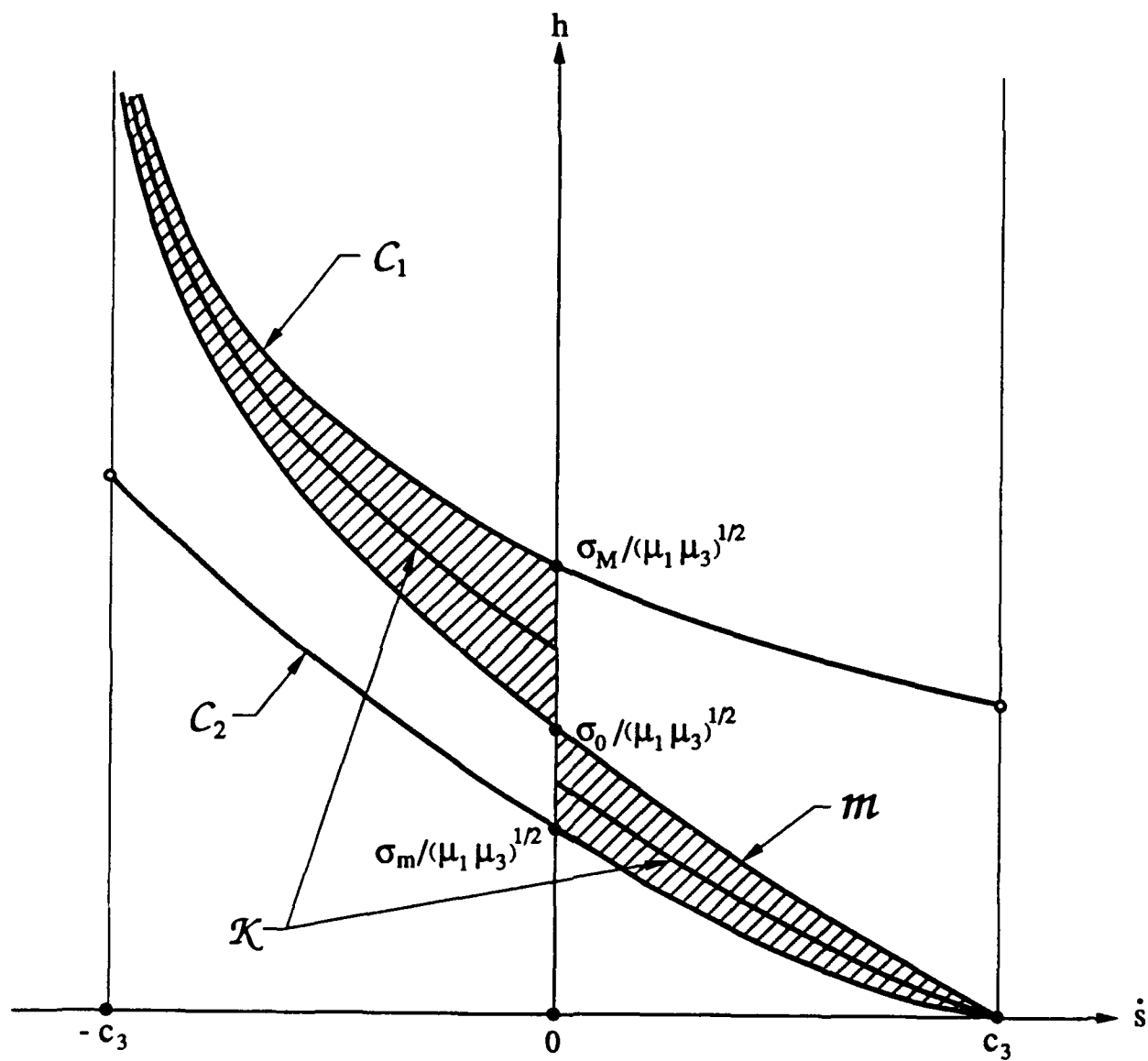
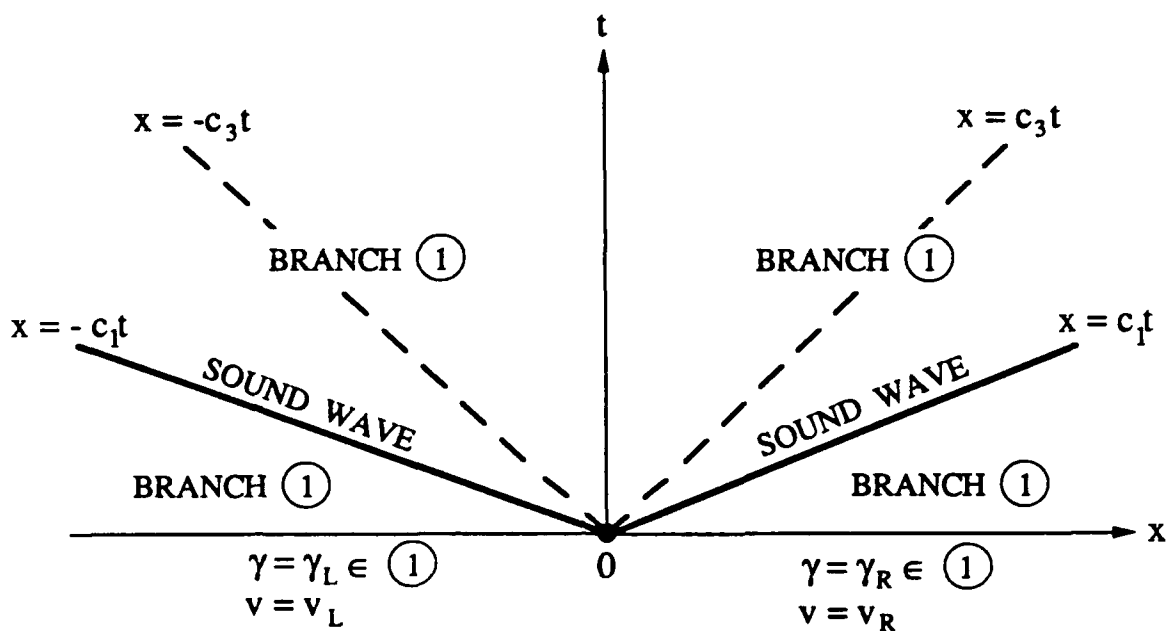
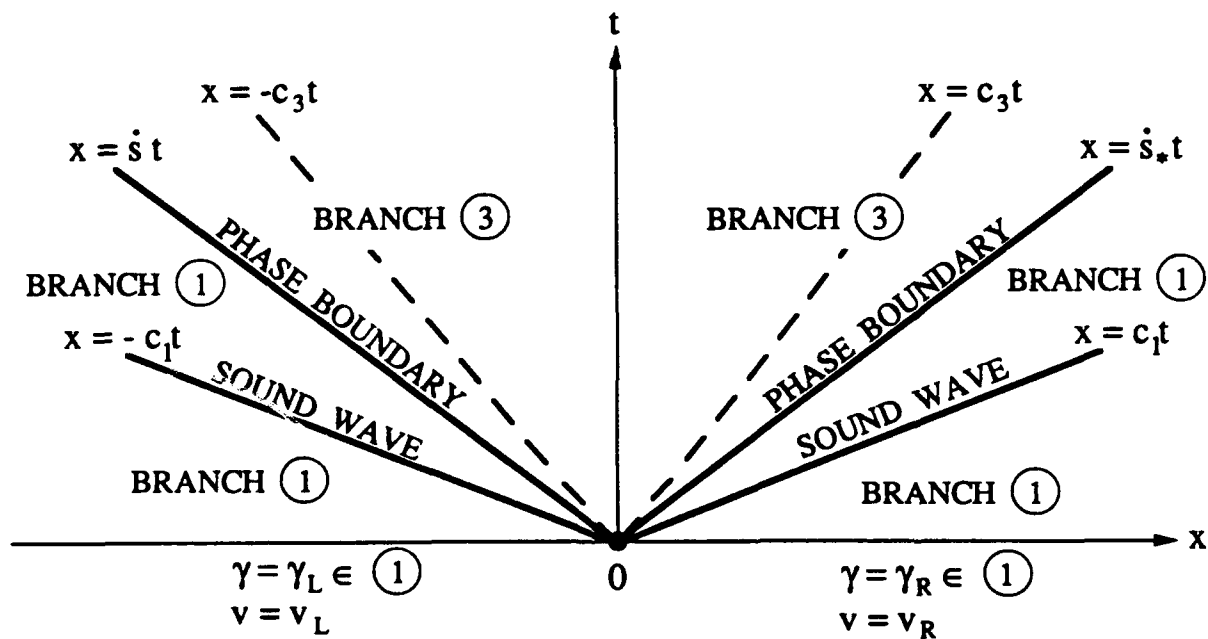


FIGURE 8. THE  $\dot{s}, h$ -PLANE FOR 1,3-INITIAL DATA .





(a) BAR REMAINS ON BRANCH ①



(b) BAR CHANGES TO BRANCH ③

FIGURE 9. FORM OF SOLUTIONS TO THE 1,1 - RIEMANN PROBLEM FOR THE TRILINEAR MATERIAL.

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